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The expansion of a plasma from a spherical source into a vacuum
Part 1. Fully-ionized flow

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Hamel & Willis (1966) first showed that anisotropy in temperature could be predicted theoretically in the expansion of a spherical source of gas into a vacuum, thus explaining anisotropies observed to occur in rarefied gas jets. Chou & Talbot (1967) carried out numerical calculations based on similaı ideas for a plasma consisting of electrons, ions and neutral particles. These showed that the electron temperature remains isotropic, and little anisotropy occurring among the ions within the (rather extensive) calculations. Here, in the first of two papers, an analytic treatment is applied to the expansion of a fully ionized plasma source, with the particles interacting according to the Coulomb law. The governing equations are derived from Poisson's equation and the full Boltzmann equation, with the assumption of an ellipsoidal distribution function, in order to close the moment equations and facilitate calculations of collision integrals. The investigation shows that the electrons and ions maintain equal isotropic temperatures, which fall adiabatically. In an appendix, a critical law of interaction for the freezing of the expansion is determined, accounting for the difference in behaviour of a plasma and a binary mixture of Maxwell molecules. The unsteady expansion is also considered, and grounds found for supporting an assertion by Grundy & Thomas (1969) that an isotropic inviscid solution is likely for the whole flow field, excluding a region near the expansion front, whatever the form of the law of interaction.

1. Introduction

The theory of the steady expansion of a neutral gas from a spherical sonic source into a vacuum was developed by Hamel & Willis (1966). It depended upon the condition that hypersonic flow be achieved in the continuum phase of the expansion, implying a sharply peaked distribution function downstream, for which a systematic approximation to the moments of the Boltzmann equation could be made. This 'hypersonic approximation' afforded a theoretical approach to the transition from continuum to rarefied flow, by means of the method of matched asymptotic expansions. The work of Hamel & Willis explained the observed anisotropy of temperature occurring in rarefied gas jets, and the 'freezing' of temperature in the direction of motion. For the spherical expansion, this is revealed by the emergence of two distinct temperatures, $T_{\parallel}$ and $T_{\perp}$, along and perpendicular to the streamlines, respectively. Cooper & Bienkowski (1966) applied the method of Hamel & Willis to a mixture of gases of comparable number density but disparate specific masses. The anisotropy and freezing of parallel temperature was found to occur for both species of gas, but first among the heavy particles. In both of these papers, the interactions were calculated for Maxwell molecules, i.e. for an inverse fifth-power law of interaction between particles.

An extension to a plasma consisting of electrons, ions and neutral particles was made by Chou & Talbot (1967). These authors included the possibility of ionization of neutral particles and recombination as the result of collisions. Their approach was numerical. In their calculations they assumed quasi-neutrality. They adopted two 'ellipsoidal Maxwellian' forms for the distribution functions, one for neutral atoms and ions and the other for electrons, thereby introducing parallel and perpendicular temperatures from the start. The divergence of these temperatures from each other during the expansion could then be studied. As far as the calculations were carried, the electron temperature remained isotropic. For low initial ionization there was found to be a large difference between electron and ion temperatures, but both remained isotropic out to 70 diameters distance from the centre, the collision frequency remaining high. For higher initial ionization there was some anisotropy in the ion temperature, but the difference between electron and ion temperature fell, and no freezing of temperature was reached in either of these calculations.

In this paper, an analytical investigation of the behaviour of a plasma is made, starting, as Chou & Talbot did, with ellipsoidal Maxwellian forms for the distribution functions. The calculation of the interaction terms (collision integrals) involves considerable labour; the results are presented in appendix A, as they are required in this, and a subsequent paper (part 2), and may be of value to other workers in the field. As a first approach, the technique of Hamel & Willis has been applied to a fully ionized plasma, with recombination neglected. It is shown that under this condition the plasma behaves always as if it were a continuum! The results obtained are consistent with those of Chou & Talbot when viewed as a possible limit of theirs for full ionization.

The continuum behaviour of the plasma was seen to be the direct result of the strong (Coulomb) interaction law, and this suggested that other interaction laws should be examined to determine at what level of interaction the freezing of temperature, characteristic of rarefied behaviour, gives way to 'continuum' expansion. This work is presented in appendix B, where it is shown that, for interaction laws given by inverse powers of the distance between the particles, continuum flow holds for $s < \frac{13}{5}$. The result would appear to apply equally to jets of Hamel & Willis type.

In §4 an unsteady flow is considered, and it is shown that the continuum expansion is valid for all interaction laws, at least to second order, in the whole flow field excluding a region near the expansion front, confirming a conjecture of Grundy & Thomas (1969).

In part 2 the role of partial ionization will be discussed, and the variations in behaviour consequent upon part-Coulomb and part-neutral-gas interactions examined.

2. Equations of motion

The basic equations for the steady flow are the Boltzmann equations for the different species of particle present. If for the sth species, $f_s \equiv f_s(\mathbf{x}, \mathbf{v})$ is the one-particle distribution function, $\mathbf{v}$ the particle velocity at the location $\mathbf{x}$, and $\mathbf{F}_s$ the external force per unit mass acting on the particle, then

$$\mathbf{v}\,.\,\nabla_{\mathbf{x}} f_s + \mathbf{F}_s\,.\,\nabla_{\mathbf{v}} f_s = \sum_{\sigma} \int (\tilde{f}_s \tilde{f}_\sigma - f_s f_\sigma)\, gb\, db\, d\epsilon D v_\sigma. \tag{2·0}$$

On the r.h.s., $\tilde{f}$ indicates the distribution function *after* elastic collisions; g is the relative speed of impacting particles. b is the impact parameter and ϵ is the orientation angle of the impact plane. Velocities occurring in the expressions for the distribution functions need to be given suffices ($\mathbf{v}_s$, $\mathbf{v}_\sigma$), in order to distinguish them where they appear under the integral signs. Dv_σ is the element of volume in the velocity-space of the σth species of particle; and the summation is over all possible σ (including $\sigma = s$). $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{v}}$ represent, as usual, the operation of taking the gradient in the $\mathbf{x}$-space and $\mathbf{v}$-space, respectively.

For each kind of particle, an ellipsoidal Maxwellian form of distribution function is assumed for the purpose of calculation of the collision integrals. This implies a diagonal form of stress tensor for each species. There is confirmation in the work of Hamel & Willis that this is suitable for the neutral jet. Legge (1970) has shown that it represents the Knudsen effusion from a slit in a plate very accurately. Freeman & Thomas (1968) have also demonstrated that this ansatz provides a good approximation for Maxwell molecules. It involves the introduction of two temperatures: for spherically symmetric flow these are $T_\parallel$ along the streamlines, and $T_\perp$ transverse to the streamlines. The one-particle distribution functions are thus taken to be

$$f_s = n_s \left(\frac{m_s}{2\pi k T_{s\parallel}}\right)^{\frac{1}{2}} \left(\frac{m_s}{2\pi k T_{s\perp}}\right) \exp\left[-\frac{m_s(v_r - u_s)^2}{2kT_{s\parallel}} - \frac{m_s(v_\theta^2 + v_\phi^2)}{2kT_{s\perp}}\right].$$

Here n_s, m_s are respectively the number density and particle mass of the sth species, while u_s is the mean (necessarily radial) speed of this species, v_r, v_θ, v_ϕ are the components in spherical polar co-ordinates of the velocity of the particle under consideration, and k is Boltzmann's constant.

In what follows, the suffices I, E will refer to ions and electrons, respectively.

The form chosen for the distribution functions automatically removes the heat-flux tensor, and permits the termination of the moment equations at the energy equation level. For a gas consisting only of ions and electrons, the collisions obey the Coulomb law. In order to obtain a finite contribution to the collision term, the limits of integration are restricted so as to include only collisions occurring within the Debye sphere. Long-range collisions are catered for by the appearance of the electric field among the forces acting on the particle, giving rise to a term on the l.h.s. of the Boltzmann equation. In consequence, the Poisson equation has to be taken into account. There are nine unknowns: the electric field E (with only a radial component), u_I, u_E, n_I, n_E, $T_{I\parallel}$, $T_{E\parallel}$, $T_{I\perp}$, $T_{E\perp}$.

These are determined from Poisson's equation together with eight moment equations derived from the Boltzmann equations. Two moments of zero order are available: these are the equations of continuity for the electrons and for the ions. Two moments of first order may be chosen corresponding to the radial momentum: those chosen represent the rate of change of the radial momentum of the electrons and the conservation of radial momentum for the plasma as a whole. Four moments of second order may be taken: the ones used represent the rate of change of energy of the electrons, the conservation of energy for the plasma as a whole, the rate of change of radial stress in the electron gas, and the same rate of change in the ion gas. The details of the derivation of the r.h.s. of these equations are given in appendix A. The results are listed below for convenience in equations (2·1)–(2·9).

All of these equations are given in non-dimensional form. The transformations effecting this are given below, with dimensional variables marked by a dash, and an asterisk denoting conditions at the sonic radius in the source (i.e. where the radius $r' = r^*$ and the Mach number $M = 1$):

$$r' = r^* r, \quad n'_I/n_I = n'_E/n_E = n^*, \quad u'_E/u_E = u'_I/u_I = u^*,$$

$$T'_{I\parallel}/T_{I\parallel} = T'_{E\parallel}/T_{E\parallel} = T^* = T'_{I\perp}/T_{I\perp} = T'_{E\perp}/T_{E\perp}.$$

Further, the Debye length h^*_D, defined by $h^{*2}_D = \epsilon_0 kT^*/n^* e^2$ (where ϵ_0 is the permittivity of free space and $-e$ is the electric charge on an electron), provides a dimensionless electric field E defined by

$$E' = E(kT^*/eh^*_D).$$

If the source quantities are taken, as in Chou & Talbot, to be

$$n^* = 0{\cdot}5 \times 10^{23}\,\mathrm{m}^{-3}, \quad r = 10^{-2}\,\mathrm{m}, \quad \text{and} \quad T^* = 10^4\,{}^\circ\mathrm{K},$$

then, for a hydrogen jet, $u^{*2} = \frac{5}{3}(p'_E + p'_I)/\rho' = \frac{10}{3}kT^*/(m_E + m_I)$, p' being pressure and ρ' the plasma density, and $u^* \simeq 1{\cdot}6 \times 10^4\,\mathrm{m\,sec^{-1}}$.

Poisson's equation: $$(1/r^2)(d/dr)(r^2 E) = \delta(n_I - n_E), \quad (2{\cdot}1)$$

where $$\delta = er^*(n^*/\epsilon_0 kT^*)^{\frac{1}{2}} \simeq 10^6.$$

Equations of continuity: $$r^2 n_E u_E = 1 = r^2 n_I u_I. \quad (2{\cdot}2), (2{\cdot}3)$$

Rate of change of radial electron momentum:

$$\gamma_E^2 n_E u_E du_E/dr + (d/dr)(n_E T_{E\parallel}) + 2n_E(T_{E\parallel} - T_{E\perp})/r + n_E E\delta$$
$$= P\lambda\gamma_E^2(u_I - u_E)\, n_I n_E(1 + 3\mu^2/5)/(T_{E\parallel})^{\frac{3}{2}}, \quad (2{\cdot}4)$$

where $$\gamma_E^2 = m_E u^{*2}/kT^* = u^{*2}/c_E^{*2},$$

with $$c_E^{*2} = kT^*/m_E,$$

$$P = (4\sqrt{(2\pi)}/3)(1 + m_E/m_I)\ln\Lambda$$

and $$\lambda = r^* n^* e^4/16\pi^2 u^* \epsilon_0^2 m_E^2 c_E^{*3},$$

γ_E^2, P and λ being dimensionless constants, and Λ a cut-off parameter. The expansion parameter μ^2 is defined by $\mu^2 = 1-(c_\perp/c_\parallel)^2$, $c_\parallel$ and $c_\perp$ being characteristic speeds $\{(2kT_{E\parallel}/m_E)+(2kT_{I\parallel}/m_I)\}^{\frac{1}{2}}$ and $\{(2kT_{E\perp}/m_E)+(2kT_{I\perp}/m_I)\}^{\frac{1}{2}}$, respectively.

Conservation of plasma momentum:

By use of the equations of continuity and Poisson's equation, this can be expressed in the form

$$\gamma_E^2 du_E/dr + \gamma_I^2 du_I/dr + r^2(d/dr)(n_I T_{I\parallel} + n_E T_{E\parallel}) + 2r(n_I T_{I\parallel} + n_E T_{E\parallel} - n_I T_{I\perp} - n_E T_{E\perp}) - E\,d(r^2E)/dr = 0, \quad (2\cdot5)$$

where $\gamma_I^2 = m_I u^{*2}/kT^* = u^{*2}/c_I^{*2}$, with $c_I^{*2} = kT^*/m_I$. By addition,

$$\gamma_E^2 + \gamma_I^2 = \tfrac{10}{3}.$$

Rate of change of electron energy:

$$(3/2)\,n_E u_E dT_E/dr - u_E T_{E\parallel} dn_E/dr - 2u_E n_E(T_{E\parallel}-T_{E\perp})/r = 3P\lambda(n_E n_I/\sqrt{T_{E\parallel}})(m_E/m_I)\,[\{(T_I/T_{E\parallel})-1\}+\mu^2\{(3T_{I\parallel}+12T_{I\perp}-5T_{E\parallel})/15T_{E\parallel}\}] \quad (2\cdot6)$$

Here, the 'resultant temperatures' T_E, T_I are used, defined by

$$3T_E = T_{E\parallel}+2T_{E\perp}, \quad 3T_I = T_{I\parallel}+2T_{I\perp}.$$

Conservation of plasma energy:

$$3(T_{E\parallel}+T_{I\parallel})+2(T_{E\perp}+T_{I\perp})+\gamma_E^2(u_E^2-1)+\gamma_I^2(u_I^2-1) = 10. \quad (2\cdot7)$$

Rate of change of electron $\hat{r}\hat{r}$-stress:

$$n_E(d/dr)\,(u_E^2 T_{E\parallel}) = -(\tfrac{4}{5})\,P\lambda u_E(n_E n_I/\sqrt{T_{E\parallel}})\{\sqrt{2}(n_E/n_I)+2\}\mu^2. \quad (2\cdot8)$$

Rate of change of ion $\hat{r}\hat{r}$-stress:

$$n_I(d/d_r)\,(u_I^2 T_{I\parallel}) = \tfrac{4}{5}P\lambda u_I(n_E n_I/\sqrt{T_{E\parallel}})\,[-\sqrt{2}(n_I/n_E)\,(m_E T_{E\parallel}/m_I T_{I\parallel})^{\frac{1}{2}}(1-T_{I\perp}/T_{I\parallel}) + (m_E/m_I)\{(1-T_{I\parallel}/T_{E\parallel})+\mu^2\}]. \quad (2\cdot9)$$

3. Solution in series

The key parameter in the solution in series of the system of equations (2·1)–(2·9) is $\lambda(m_E/m_I)^{\frac{1}{2}}$. Let this parameter be written as ν. In view of the definition of λ, it may be regarded as a dimensionless source collision frequency for close Coulomb encounters. It is large, being approximately 800 foı a hydrogen source. Of the other dimensionless constants occurring in the equations, δ, $P\lambda$, $P\lambda(m_E/m_I)^{\frac{1}{2}}$ are $O(\nu^2)$, $P\lambda(m_E/m_I)$ and $P\lambda\gamma_E^2$ are $O(\nu)$, γ_I^2 is $O(1)$, while γ_E^2 is $O(\nu^{-1})$. The expansion parameter μ^2 is zero under isotropic conditions. Write

$$n_I = n_{I0}n_0 + n_{I1}\nu^{-1} + n_{I2}\nu^{-2}+\ldots,$$

and similar expressions for all the other unknowns. Terms $O(\nu^2)$ in (2·1) give immediately that $n_{I0} = n_{E0}$, while, from (2·2) and (2·3), $u_{I0} = u_{E0}$. From (2·4), $E_0 = 0$. From (2·8), $\mu_0 = 0$, whence $(T_{E\parallel})_0 = (T_{E\perp})_0 = T_{E0}$, say. Then, from (2·9),

$(T_{I\parallel})_0 = (T_{I\perp})_0 = T_{I0}$, say. Terms $O(\nu)$ in (2·5) give $T_{I0} = T_{E0} = T_0$, say. Then, from (2·1)–(2·4), it follows that

$$n_{I1} = n_{E1}, u_{I1} = u_{E1}, E_1 = 0, (T_{E\parallel})_1 = (T_{E\perp})_1 = T_{E1}, \quad \text{say,}$$

and $$(T_{I\parallel})_1 = (T_{I\perp})_1 = T_{I1}, \quad \text{say.}$$

The first approximation to the energy equation is obtained by equating terms $O(1)$ in (2·7). Thus,

$$\left.\begin{aligned} 10T_0 + (\gamma_E^2 + \gamma_I^2)(u_0^2 - 1) &= 10, \\ T_0 &= \tfrac{1}{3}(4 - u_0^2). \end{aligned}\right\} \tag{3·1}$$

whence

Equation (2·6) yields a differential equation relating temperature and velocity

$$\tfrac{5}{3} du_0/dr + r^2 (d/dr)(T_0/u_0 r^2) = 0.$$

By use of (3·1), this can be integrated to yield the result,

$$u_0(4 - u_0^2)^{\frac{3}{2}} r^2 = 3^{\frac{3}{2}}, \tag{3·2}$$

on applying the initial condition $u_0 = 1$ when $r = 1$. This may be written by virtue of (3·1) in the alternative form

$$u_0^2 T_0^3 = 1/r^4, \tag{3·3}$$

true for all r. As $r \to \infty$, $u_0 \sim 2 - 3/16^{2/3} r^{4/3} + \dots$ and $T_0 \sim \tfrac{1}{4} r^{-4/3} + \dots$.

Equation (2·5) gives the differential equation

$$\tfrac{3}{2} n_0 u_0 dT_0/dr - u_0 T_0 dn_0/dr = 3P\lambda(m_E/m_I)(n_0^2/T_0^{\frac{3}{2}})(T_{I1} - T_{E1}).$$

On division by $u_0 T_0^{\frac{5}{2}}$, and use of the equation of continuity, this may be expressed in the form,

$$-(d/dr)(1/u_0 r^2 T_0^{\frac{3}{2}}) = 3P\lambda(m_E/m_I)(n_0^2/u_0 T_0^4)(T_{I1} - T_{E1}).$$

In view of (3·3) this shows that $T_{I1} = T_{E1} = T_1$, say. Thus, the isotropic condition on the temperature holds up to this order of approximation.

Equations (2·1)–(2·3) yield $n_{I2} = n_{E2}, u_{I2} = u_{E2}$, and (2·4) leads to

$$E_2 = -(5/2\delta)(dT_0/dr).$$

Thus, the diffusion velocity vanishes to second order of smallness, the plasma is quasi-neutral to the same order, and the electric field is very small indeed. The parameter μ^2 is $O(\nu^{-2})$ and from (2·8) it follows that

$$(T_{E\parallel})_2 - (T_{E\perp})_2 = O(r^{-\frac{7}{3}}), \quad \text{for large } r, \tag{3·4}$$

and a similar result is obtained from (2·9) for the anisotropy in the ion temperature. The pattern is now set up, although the assumptions inherent in the derivation of the basic equations naturally limit the extent to which the process may be continued. The departure from isotropy is small compared with the leading term in the isentropic solution $O(r^{-\frac{4}{3}})$, and it thus appears that the non-uniformity in the series expansion, which accounted for the freezing of radial temperature in the Hamel & Willis jet and in the Cooper & Bienkowski mixture, either does not occur in the fully ionized plasma jet, or at least is very long delayed indeed.

This behaviour, which is at first rather surprising, might perhaps have been anticipated from a result in the kinetic theory of gases, which states that, under the condition of thermodynamic equilibrium, the mean free path $\lambda \propto T^{2/(s-1)}/n$, when the interaction law in a gas containing just one type of particle follows an inverse sth power law. For $n \propto r^{-2}$ and $T \propto r^{-\frac{4}{3}}$, and the Coulomb law ($s = 2$), as in the rarefied jet foı $r \gg 1$, $\lambda \propto r^{-\frac{2}{3}}$, i.e. $\lambda \ll 1$ for $r \gg 1$. The large collision cross-section causes the gas to behave like a continuum, even at large distances from the source. Breakdown of continuum flow might be expected when $\lambda \to \infty$ as $r \to \infty$. This would be the case, from the above formula with general values of s, when $s > \frac{7}{3}$. In appendix B this question is examined for flows described by an elliptic distribution function and the critical power of s is found to be $s = \frac{13}{5}$. Thus, among integral-power laws ($s > 1$), the Coulomb law stands alone.

4. Unsteady expansion

The Boltzmann equation for unsteady, spherically symmetric flow has one more term on the l.h.s. than (2·0). This extra term is $\partial f/\partial t$. It has the result of introducing new terms into the l.h.s. of the moment equations, but has no effect on the form of the r.h.s. Poisson's equation remains unchanged:

$$(1/r^2)(d/dr)(r^2E) = \delta(n_I - n_E). \tag{4·1}$$

The equations of continuity become

$$\partial n_E/\partial t + r^{-2}(\partial/\partial r)(r^2 n_E u_E) = 0, \tag{4·2}$$

and

$$\partial n_I/\partial t + r^{-2}(\partial/\partial r)(r^2 n_I u_I) = 0, \tag{4·3}$$

where t has been non-dimensionalized with r^*/u^*.

In many papers, one of these equations is replaced by the condition for zero current, namely $n_E u_E = n_I u_I$. This will be shown to be valid to a high degree of approximation by means of the series expansion, but will not be assumed to be true. The radial electron momentum equation (4·4), not written out, is changed only by the addition of a term $\gamma_E^2 n_E \partial u_E/\partial t$ to the l.h.s. of (2·4). The l.h.s. of the equation of conservation of plasma momentum is changed, because of the lack of an integral of the equations of continuity. This equation now becomes

$$\gamma_E^2 n_E Du_E/Dt + \gamma_I^2 n_I Du_I/Dt + (\partial/\partial r)(n_I T_{I\parallel} + n_E T_{E\parallel}) + (2/r)(n_I T_{I\parallel} + n_E T_{E\parallel} - n_I T_{I\perp} - n_E T_{E\perp}) - r^{-2}E(\partial/\partial r)(r^2 E) = 0, \tag{4·5}$$

where $D/Dt = \partial/\partial t + u\partial/\partial r$. The electron energy equation is now

$$\tfrac{3}{2} n_E DT_E/Dt + n_E T_{E\parallel} \partial u_E/\partial r + 2n_E T_{E\perp} u_E/r = \text{r.h.s. of } (2·6). \tag{4·6}$$

The equation of conservation of plasma energy, no longer integrable, is

$$n_E(D/Dt)(3T_E + \gamma_E^2 u_E^2) + n_I(D/Dt)(3T_I + \gamma_I^2 u_I^2) + 2r^{-2}(\partial/\partial r)(r^2 n_E u_E T_{E\parallel} + r^2 n_I u_I T_{I\parallel}) = 0. \tag{4·7}$$

Equations (2·8) and (2·9) are modified to (4·8), (4·9), also not written out, by the addition of terms $n_E u_E \partial T_{E\parallel}/\partial t$, $n_I u_I \partial T_{I\parallel}/\partial t$ to their l.h.s., respectively.

The expansion in series is carried out exactly as in the steady case. Equation (4·1) gives $n_{E0} = n_{I0} = n_0$. Then, from (4·2) and (4·3), by subtraction,

$$\frac{\partial}{\partial r}\{r^2 n_0(u_{I0} - u_{E0})\} = 0,$$

i.e.

$$r^2 n_0(u_{I0} - u_{E0}) = F(t),$$

where $F(t)$ is a function of time t to be determined. For a continuum source starting to expand with constant flux, the electron and ion fluxes are equal at $r = 0$, and hence $F(t) = 0$. If, on the other hand, the expansion derives from a fixed mass of gas expanding into a vacuum, then, on the vacuum front $r = R(t)$, the density n_0 must vanish for every t while the velocity remains finite; hence, $F(t) = 0$ in this case also. Thus, $u_{E0} = u_{I0} = u_0$, say. Then (4·4) gives $E_0 = 0$, and, from (4·6), (4·8) and (4·9), $(T_{E\parallel})_0 = (T_{E\perp})_0 = (T_{I\parallel})_0 = (T_{I\perp})_0 = T_0$, say, and equations (4·3), (4·5) and (4·7) yield the following set of equations relating u_0, n_0 and T_0:

$$\partial n_0/\partial t + r^{-2}(\partial/\partial r)(r^2 n_0 u_0) = 0, \qquad (4·10)$$

$$5n_0(\partial u_0/\partial t + u_0 \partial u_0/\partial r) + 3(\partial/\partial r)(n_0 T_0) = 0, \qquad (4·11)$$

$$n_0(\partial/\partial t + u_0 \partial/\partial r)(9T_0 + 5u_0^2) + r^{-2}(\partial/\partial r)(6r^2 n_0 u_0 T_0) = 0. \qquad (4·12)$$

A self-similar solution of these equations has been given by Keller (1956). It describes the motion of a fixed mass of gas, released into a vacuum at time $t = 0$ after being contained between $r = 0$ and $r = 1$ with $n_0 = 1$ at $r = 0$ and $n_0 = 0$ at $r = 1$. If $r = R(t)$ defines the position of the front of the expansion at time t, the solution is

$$R = (3t^2+1)^{\frac{1}{2}}, \quad u_0 = (r/R)(dR/dt) = 3rt/(3t^2+1),$$

$$n_0 = R^{-6}(R^2 - r^2)^{\frac{3}{2}}, \quad T_0 = (R^2 - r^2)/R^4.$$

For large r and t, but such that r/t remains finite, this solution becomes

$$R \sim \sqrt{3}t, \quad u_0 \sim r/t, \quad n_0 \sim (3t^2 - r^2)^{\frac{3}{2}}/27t^6, \quad T_0 \sim (3t^2 - r^2)/9t^4.$$

On equating terms of order ν in the moment equation (4·1), and ν^{-1} in (4·2), it is seen that $n_{I1} = n_{E1} = n_1$, say, and $u_{E1} = u_{I1} = u_1$, say, in the same way as for the terms of lower order. Equations (4·8) and (4·9) show that

$$(T_{E\parallel})_1 = (T_{E\perp})_1 = T_{E1}, \text{ say}, \quad (T_{I\parallel})_1 = (T_{I\perp})_1 = T_{I1}, \text{ say}.$$

From (4·4), $E_1 = 0$. In (4·6) the terms $O(1)$ on the l.h.s. involve only n_0, u_0, T_0, and their sum vanishes identically on substitution. Hence, from the r.h.s., $T_{E1} = T_{I1} = T_1$, say.

Equations (4·1)–(4·3) yield $n_{I2} = n_{E2}$, and hence $u_{I2} = u_{E2}$. From (4·8) and (4·9), again because n_0, u_0, T_0 make the l.h.s. vanish identically, $(T_{E\parallel})_2 = (T_{E\perp})_2$ and $(T_{I\parallel})_2 = (T_{I\perp})_2$. Furthermore, $(T_E)_2 = (T_I)_2$, this result following from the fact that the l.h.s. of (4·6) vanishes identically when the equations for u_1, n_1 and T_1 are used. The isotropic, inviscid solution therefore remains valid up to and including the third approximation, preserving quasi-neutrality and justifying the assumption usually made that no current flows ($n_E u_E = n_I u_I$).

The behaviour of u_1, n_1, and T_1 for large r and t, with r/t remaining finite, is derived from (4·3), (4·5) and (4·7) by using a similar solution again. This gives

$$T_1 \propto t/(3t^2-r^2)^{\frac{3}{2}}, \quad u_1 \propto t^4/r^2(3t^2-r^2)^{\frac{3}{2}}, \quad n_1 \propto 1/t(3t^2-r^2).$$

On considering the ratios

$$T_1/T_0 \propto t^5/(3t^2-r^2)^{\frac{5}{2}}, \quad u_1/u_0 \propto t^5/r^3(3t^2-r^2)^{\frac{3}{2}} \quad n_1/n_0 \propto t^5/(3t^2-r^2)^{\frac{5}{2}},$$

it is clear that, as far as the second-order terms are concerned, the expansion in ν^{-1} remains valid until very close to the front, but as r tends to $R(1-\frac{1}{2}\nu^{-\frac{2}{5}})$, νT_1 becomes of similar size to T_0 and the expansion cannot be continued for larger r ($\nu^{-\frac{2}{5}} \simeq 0{\cdot}06$).

The ratio of the Debye length to the region size behaves as $1/(R-r)^{\frac{1}{2}}$, so large electrical effects and charge separation are expected as the front is approached. This is consistent with the acceleration of ions by electrons, noticed by Plyutto (1967), which produces a narrow double layer at the surface of an expanding plasma drop containing ions of very high energies and with a high electric field.

A similar validity of the isotropic, inviscid solution for the whole flow field, excluding the insolvable region near the front, is found by Grundy & Thomas (1969) in their investigation of the unsteady spherical expansion of a fixed mass of gas consisting of Maxwell molecules. This is in marked contrast to the steady case, and to the unsteady cylindrical expansion for Maxwell molecules studied by Freeman & Grundy (1968). Grundy & Thomas remark that their solution is likely to apply to all forms of molecular interaction.

It is clear that the results obtained above for unsteady flow support this assertion. The equality of the temperatures T_0, T_1 and T_2 derives solely from the l.h.s. of the moment equations, and is not dependent on the form of the collision term.

Mrs Goldfinch is indebted to the Science Research Council for a Research Studentship, which enabled her to take part in this work. The general results of this paper were described at the Euromech Colloquium no. 13 held at the National Physical Laboratory, Teddington, in July 1969.

Appendix A. Moments of the collision term for a plasma with ellipsoidal distribution functions

The collision term for the single electron distribution function is

$$\int (\tilde{f}_E \tilde{f}_I - \tilde{f}_E f_I)\, gb\, db\, d\epsilon\, Dv_I.$$

For Coulomb attractions (see e.g. Chapman & Cowling 1960, p. 177 f.),

$$gb\,db = \frac{-e^4(m_E+m_I)^2 \cos\frac{1}{2}\chi}{32\pi^2 m_E^2 m_I^2 g^3 \epsilon_0^2 \sin^3\frac{1}{2}\chi}\, d\chi, \tag{A 1}$$

where χ is the angle of the deflexion for a particle of relative speed g and impact parameter b.

The gain in momentum of the electrons from collisions occurs only as the result of encounters with ions, on account of the conservation law for collisions among like particles. It is therefore represented by

$$\mathbf{I} = m_E \int \mathbf{v}_E(\tilde{f}_E \tilde{f}_I - f_E f_I)\, gb\, db\, d\epsilon\, Dv_I\, Dv_E$$

$$= m_E \int (\tilde{\mathbf{v}}_E - \mathbf{v}_E) f_E f_I gb\, db\, d\epsilon\, Dv_I\, Dv_E,$$

by a well-known transformation. The conservation of momentum in the encounter gives the equation

$$m_E \tilde{\mathbf{v}}_E + m_I \tilde{\mathbf{v}}_I = m_E \mathbf{v}_E + m_I \mathbf{v}_I,$$

whence $$\tilde{\mathbf{v}}_E - \mathbf{v}_E = \{m_I/(m_I + m_E)\}(\tilde{\mathbf{g}}_{EI} - \mathbf{g}_{EI}),$$

where $\mathbf{g}_{EI}$ is the velocity of the electron relative to the ion. If $\hat{\boldsymbol{\beta}}$ is a unit vector perpendicular to $\mathbf{g}_{EI}$, then, on writing $\mathbf{g}$ for $\mathbf{g}_{EI}$,

$$\tilde{\mathbf{v}}_E - \mathbf{v}_E = \{m_I/(m_E + m_I)\}(\mathbf{g}\cos\chi - \mathbf{g} + \hat{\boldsymbol{\beta}} g \sin\chi).$$

Writing $\hat{\boldsymbol{\beta}} = \hat{\boldsymbol{\alpha}}\cos\epsilon + \boldsymbol{\gamma}\sin\epsilon$, $\hat{\boldsymbol{\alpha}}$ and $\boldsymbol{\gamma}$ being unit vectors respectively in and perpendicular to the plane $\epsilon = 0$ and lying in the plane perpendicular to $\mathbf{g}$, we can perform the ϵ-integrations very simply, to obtain

$$\mathbf{I} = +\frac{e^4(m_E + m_I)}{8\pi\epsilon_0^2 m_E m_I} \int f_E f_I \frac{\mathbf{g}}{g^3} \cot\tfrac{1}{2}\chi\, d\chi\, Dv_I\, Dv_E. \quad \text{(A 2)}$$

As shown in appendix B, for the Coulomb law of interaction the integral must be cut off in order to yield a finite result; it is customary to apply an upper limit corresponding to interactions at the distance of the Debye length for which $\chi = \chi_D$, $\tan\frac{1}{2}\chi_D = -1/\Lambda$, say. Then

$$\int_{-\pi}^{-\chi_D} \cot\tfrac{1}{2}\chi\, d\chi = -\ln(1 + \Lambda^2) \simeq -2\ln\Lambda \quad (\Lambda \gg 1),$$

whence $$I = -\frac{e^4(m_E + m_I)}{4\pi\epsilon_0^2 m_E m_I} \ln\Lambda \int f_E f_I \frac{\mathbf{g}}{g^3} Dv_I\, Dv_E.$$

The terms neglected in the logarithm are of order $1/(2\Lambda^2 \ln\Lambda)$ times those retained. The integral $\mathbf{I}$ is a particular case of the general momentum integral discussed in appendix B.

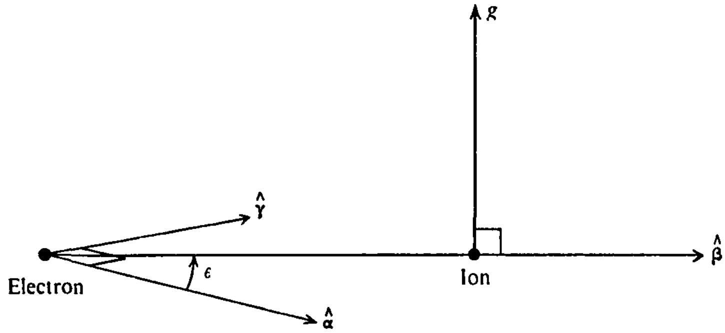

FIGURE 1

The moment of the Boltzmann equation taken with $\phi = \frac{1}{2}m_E(v_E - u_E)^2$, which is required to obtain the energy equation for the electrons, may be obtained as the spur of the tensor during the evaluation of the contributions from individual terms required for the equation of electron $\hat{r}\hat{r}$-stress.

The general moment of the Boltzmann equation for the electron distribution function after multiplication by ϕ, and integration over the velocity space of the electrons, may be written as

$$\frac{\partial}{\partial t} N_E\langle\phi\rangle + \nabla . \{N_E\langle\phi \mathbf{v}_E\rangle\} - N_E\left\langle\frac{\partial\phi}{\partial t} + \mathbf{v}_E . \nabla_{\mathbf{x}}\phi + \mathbf{F}_E . \nabla_{\mathbf{v}_E}\phi\right\rangle = N_E\Delta\langle\phi\rangle,$$

where
$$N_E\langle\psi\rangle = \int \psi f_E Dv_E,$$

and
$$N_E\Delta\langle\phi\rangle = \Sigma_B \int \phi(\tilde{f}_E\tilde{f}_B - f_E f_B)\, gb\, db\, d\epsilon\, Dv_B Dv_E,$$

where B refers to electrons and/or ions as appropriate.

Let $V = v - u$ be the 'peculiar' velocity of a particle. Put

$$\phi = \tfrac{1}{2}m_E V_{E1}^2 = \tfrac{1}{2}m_E(v_{EI} - u_E)^2,$$

and identify V_{E1} with the radial component in whatever system of axes is used. In the present steady flow, with this value of ϕ, the moment equation is

$$\operatorname{div}\{\tfrac{1}{2}m_E N_E\langle v_E V_{E1}^2\rangle\} + m_E N_E\left\langle v_{E1}(v_{E1} - u_E)\frac{du_E}{dr}\right\rangle + N_E\langle eE_E V_{E1}\rangle = N_E\Delta\langle\tfrac{1}{2}m_E V_{E1}^2\rangle.$$

Now,
$$N_E\langle v_E V_{E1}^2\rangle = \int (u_E + V_{E1})\, V_{E1}^2 f_E DV_E = N_E u_E kT_{\parallel E},$$

the third-order moment vanishing, and

$$N_E\langle v_{E1}V_{E1}\rangle = N_E\langle V_{E1}^2\rangle + N_E u_E\langle V_{E1}\rangle,$$

the last term vanishing, because $\langle V_{E1}\rangle = 0$ by definition of the mean velocity u_E. This also removes the term involving the electric field. Thus, the equation for electron $\hat{r}\hat{r}$-stress is

$$\frac{1}{r^2}.\frac{d}{dr}(\tfrac{1}{2}N_E u_E r^2 kT_{\parallel E}) + N_E kT_{\parallel E}\frac{du_E}{dr} = N_E\Delta\langle\tfrac{1}{2}m_E V_{E1}^2\rangle;$$

the equation of continuity may be used to simplify the first term, and a slight rearrangement after multiplication by u_E leads to the form,

$$kN_E\frac{d}{dr}(u_E^2 T_{\parallel E}) = N_E u_E\Delta\langle m_E V_{E1}^2\rangle.$$

From collision dynamics,

$$\tilde{\mathbf{V}}_E = \mathbf{V}_E + M_I\mathbf{g}(\cos\chi - 1) + M_I\boldsymbol{\beta} g\sin\chi, \tag{A 3}$$

where $M_I = m_I/(m_I + m_E)$, and $\boldsymbol{\beta}$ is a unit vector perpendicular to $\mathbf{g}$ in the plane of the relative motion.

Let $\boldsymbol{\alpha}$ be an arbitrary vector in the plane $\epsilon = 0$. Then

$$g^2(\boldsymbol{\alpha}\,.\,\boldsymbol{\beta})^2 = \{g^2\boldsymbol{\alpha}^2 - (\mathbf{g}\,.\,\boldsymbol{\alpha})^2\}\cos^2\epsilon. \tag{A 4}$$

We use the relation,

$$m_e\int(\tilde{V}_{Ej}\tilde{V}_{Ek} - V_{Ej}V_{Ek})f_E f_I Dv_I Dv_E = \frac{\partial^2 \mathscr{I}_{EI}}{\partial\alpha_j\partial\alpha_k},$$

where $$\mathscr{I}_{EI} = \frac{m_e}{2}\int\{(\boldsymbol{\alpha}\,.\,\tilde{\mathbf{V}}_E)^2 - (\boldsymbol{\alpha}\,.\,\mathbf{V}_E)^2\}f_E f_I g b\, db\, d\epsilon\, DV_I DV_E.$$

The integrations with respect to ϵ are very simple to carry out; on substitution from (3), (4), and using (1), we derive

$$m_E\int\{(\boldsymbol{\alpha}\,.\,\tilde{\mathbf{V}}_E)^2 - (\boldsymbol{\alpha}\,.\,\mathbf{V}_E)^2\}g b\, db\, d\epsilon$$
$$= \frac{e^4(m_E+m_I)}{16\pi m_E m_I \epsilon_0^2 g^3}\int_{-\pi}^{-\chi_D}\cot\tfrac{1}{2}\chi\{4(\boldsymbol{\alpha}\,.\,\mathbf{V}_E)(\boldsymbol{\alpha}\,.\,\mathbf{g})$$
$$+ M_I(\mathbf{g}\,.\,\boldsymbol{\alpha})^2(3\cos\chi - 1) - 2M_I\cos^2\tfrac{1}{2}\chi\, g^2\boldsymbol{\alpha}^2\}\, d\chi.$$

On making the same use of the fact that $\Lambda \gg 1$ as before, we find that

$$\mathscr{I}_{EI} = \frac{e^4(m_E+m_I)}{16\pi m_E m_I \epsilon_0^2}\int_{-\infty}^{\infty}\{-8(\boldsymbol{\alpha}\,.\,\mathbf{V}_E)(\boldsymbol{\alpha}\,.\,\mathbf{g})\ln\Lambda$$
$$+ M_I(\mathbf{g}\,.\,\boldsymbol{\alpha})^2(6 - 4\ln\Lambda) + (-2 + 4\ln\Lambda)M_I g^2\boldsymbol{\alpha}^2\}\left(\frac{f_E f_I}{g^3}\right)Dv_I Dv_E.$$

The contribution to the electron energy equation is given by

$$(\nabla_\alpha\,.\,\nabla_\alpha)\mathscr{I}_{EI} = -\frac{\ln\Lambda\, e^4(m_E+m_I)}{2\pi m_E m_I \epsilon_0^2}\int_{-\infty}^{\infty}\frac{(\mathbf{V}_E\cdot\mathbf{g} - M_I g^2)}{g^3}f_E f_I DV_I DV_E$$

(the electron-electron collisions contribute nothing). The contribution to the $\hat{r}\hat{r}$-stress equation is

$$\frac{\partial^2}{\partial\alpha_1^2}\mathscr{I}_{EI} = \frac{e^4(m_E+m_I)}{16\pi m_E m_I \epsilon_0^2}\int_{-\infty}^{\infty}\{-8\ln\Lambda\, V_{E1}g_1 + (6 - 4\ln\Lambda)M_I g_1^2$$
$$+ (4\ln\Lambda - 2)M_I g^2\}\frac{f_E f_I}{g^3}DV_I DV_E.$$

It transpires, after the collection of terms not involving $\ln\Lambda$, that they start at an order of magnitude of not more than $\mu^2/\ln\Lambda$ of those retained, and they may therefore be neglected. The technique for the evaluation of these integrals under the assumption that f_E, f_I are of ellipsoidal form is indicated in the appendix B.

For the electron $\hat{r}\hat{r}$-stress equation, there is a term arising from electron-electron interaction, since the energy is partitionable between the different modes, $\hat{r}\hat{r}$, etc. For this interaction we may interpret M_I as equal to $\frac{1}{2}$, and put U, the relative mean speed of the interacting species, equal to zero. (U occurs in the transformation of the integral (A 2) above; see appendix B.) Similar considerations apply to the calculation of the terms in the ion $\hat{r}\hat{r}$-stress equation (2·9).

Appendix B. The effect of the law of intermolecular force on the validity of the continuum solution for the expansion from a spherical source

From the classical dynamics of an encounter under an intermolecular force $\mu/r^s (s > 2)$, the deflexion χ of a particle moving with relative velocity $\mathbf{g}$, and impact parameter b, is given by (Chapman & Cowling 1960, p. 177 f.)

$$\tfrac{1}{2}(\pi-\chi) = \int_0^{\beta_0} \{1-\rho^2 - 8\mu(\rho/b)^{s-1}/(s-1)\,g^2\}^{-\frac{1}{2}}\,d\rho,$$

where $\rho = b/r$, and $\beta_0 = b/r_{\min}$, r being the distance of the particle from the centre of force, and $r_{\min}$ the apsidal distance (see figure 2). From the equation of conservation of energy,

$$1-\beta_0^2 = 8\mu\beta_0^{s-1}/(s-1)\,g^2b^{s-1},$$

so that
$$g^2b^{s-1} = G(\beta_0) \quad \text{or} \quad b = (G/g^2)^{1/(s-1)}.$$

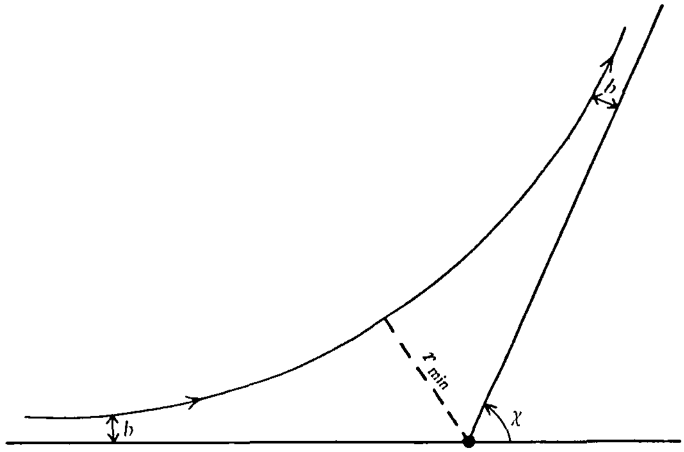

FIGURE 2

Thus,
$$\tfrac{1}{2}(\pi-\chi) = \int_0^{\beta_0} \{1-\rho^2-(1-\beta_0^2)(\rho/\beta_0)^{s-1}\}^{-\frac{1}{2}}\,d\rho = \tfrac{1}{2}h(\beta_0, s), \quad \text{say.}$$

It follows by simple calculus that $h(0,s) = 0$ and $h(1,s) = \pi$. The orientation parameters enter into the collision integral through the element $gb\,db\,d\epsilon = B(\chi,g)\,d\chi\,d\epsilon$, say. Since

$$B(\chi,g) = gb\left|(\partial b/\partial\chi)_g\right| = 2gb\left|\left(\frac{\partial b}{\partial\beta_0}\right)_g \Big/ \left(\frac{dh}{d\beta_0}\right)\right|$$

we may write finally,

$$B(\chi,g) = \frac{2G^{(3-s)/(s-1)}\,G'}{(s-1)\,(dh/d\beta_0)\,g^{(5-s)/(s-1)}}$$

$$= \left\{\frac{2}{s-1}\right\}\left\{\frac{8\mu}{s-1}\right\}^{2/(s-1)} \frac{[\beta_0\{(s-1)+(3-s)\beta_0^2\}]}{(1-\beta_0^2)^{(s+1)/(s-1)}\,(dh/d\beta_0)\,g^{(5-s)/(s-1)}}.$$

Typical of the integrals that appear among the moments is the momentum integral,

$$\mathbf{I} = \int (\tilde{\mathbf{v}}_A - \mathbf{v}_A) f_A f_B g b \, db \, d\epsilon \, Dv_B \, Dv_A,$$

where $\tilde{\mathbf{v}}_A$, $\mathbf{v}_A$ are the velocity vectors of a particle with the distribution function f_A, respectively after and before its collision with a particle with velocity $\mathbf{v}_B$ and distribution function f_B. Dv_A and Dv_B represent elements of the volume in the velocity spaces of the two particles. This integral may be converted by means of the foregoing results into

$$\mathbf{I} \propto F(s,\mu) \int \frac{f_A f_B \mathbf{g}}{g^{(5-s)/(s-1)}} Dv_B \, Dv_A,$$

with
$$F(s,\mu) = \int_0^1 \frac{\beta_0\{(s-1)+(3-s)\beta_0^2\}}{(1-\beta_0^2)^{(s+1)/(s-1)}} \cos^2\{\tfrac{1}{2}h(\beta_0,s)\}\, d\beta_0.$$

For $\beta_0 = 1-\epsilon$, $\pi - h(\beta_0, s) \propto \epsilon$. The proof is simple for $s = 2$ or 3, and the general result for $2 \leqslant s \leqslant 3$ follows from the fact that $h(\beta_0, s)$ is a monotonic decreasing function of s. Thus, $\cos^2\{\frac{1}{2}h(\beta_0,s)\} \propto \epsilon^2$ for $\beta_0 = 1-\epsilon$, $\epsilon \to 0$. The factor $F(s,\mu)$ converges, given that $s > 2$. This result is not restricted to $s \leqslant 3$; it is well-known in the kinetic theory of gases. For $s = 2$ restriction of the upper limit of the integral is required; this has been achieved here by the usual device of cutting-off the integral at collisions with impact parameter equal to the Debye length.

The second factor in the integral I is calculated by substituting for f_A and f_B, and the introduction of new variables,

$$x_i = v_{Ai} - v_{Bi}, \quad y_i = v_{Ai} + v_{Bi}.$$

We consider the component I_1. In the integral with the new variables the squares in y_1, y_2, y_3 are completed; then a translation of the origin of y co-ordinates is effected. This does not alter the limits, since these are infinite. In this way, after some algebra, we can reduce the integral to a three-dimensional one. Thus,

$$I_1 \propto \iiint_{-\infty}^{\infty} \frac{x_1 \exp[-\{(x_1-U)^2/c_\parallel^2\} - (x_2^2+x_3^2)/c_\perp^2]}{(x_1^2+x_2^2+x_3^2)^{(5-s)/2(s-1)}} dx_1\, dx_2\, dx_3,$$

where $U = u_A - u_B$, the relative mean speed of the two species of particles. Since U is always small, we may develop I_1 in powers of U by means of Maclaurin's expansion. This leads to a consideration of integrals like

$$J_1 = (U/c_\parallel^2) \iiint_{-\infty}^{\infty} \frac{x_1^2 \exp[-(x_1^2/c_\parallel^2) - (x_2^2+x_3^2)/c_\perp^2]}{(x_1^2+x_2^2+x_3^2)^{(5-s)/2(s-1)}} dx_1\, dx_2\, dx_3.$$

An examination of the expressions occurring in appendix A shows that this is the integral that determines the behaviour of the r.h.s. of the equations, so long as U^2 is negligible. In cylindrical polars this becomes

$$J_1 = \left(\frac{4\pi U}{c_\parallel^2}\right) \iint_0^{\infty} \frac{X^2 \exp[-(X^2/c_\parallel^2) - (R^2/c_\perp^2)]}{(X^2+R^2)^{(5-s)/2(s-1)}} R\, dR\, dX$$

$$= \left(\frac{\pi U}{c_\parallel^2}\right) \iint_0^{\infty} \frac{\sqrt{Y} \exp[-(Y/c_\parallel^2) - (Z/c_\perp^2)]}{(Y+Z)^{(5-s)/2(s-1)}} dY\, dZ,$$

on writing $X^2 = Y$ and $R^2 = Z$.

If we now use the well-known result

$$\int_0^\infty t^{a-1}e^{-pt}\,dt = \Gamma(a)\,p^{-a}\begin{Bmatrix}\mathrm{Re}\,a>0\\ \mathrm{Re}\,p>0\end{Bmatrix}, \tag{B 1}$$

we obtain

$$J_1 \propto \frac{1}{\Gamma[(5-s)/2(s-1)]}\iiint_0^\infty \sqrt{Y}\,t^{(7-3s)/2(s-1)}\exp\left[-Y\left(t+\frac{1}{c_\parallel^2}\right)\right.$$

$$\left.-Z\left(t+\frac{1}{c_\perp^2}\right)\right]dt\,dY\,dZ$$

$$= \frac{\Gamma(\frac{3}{2})}{\Gamma[(5-s)/2(s-1)]}\int_0^\infty \frac{t^{(7-3s)/2(s-1)}}{(t+1/c_\parallel^2)^{\frac{3}{2}}\,(t+1/c_\perp^2)}\,dt,$$

on a further double application of (B 1). The transformations require $\frac{5}{2} > (5-s)/2(s-1) > 0$ or $5 > s > \frac{5}{3}$.

This last integral can be expressed in terms of confluent hypergeometric functions. We obtain finally

$$I_1 \propto [Uc_\parallel/c_\perp^{(7-3s)/(s-1)}]\,{}_2F_1\left(\frac{3}{2},\,\frac{5-s}{2(s-1)};\,\frac{5}{2};\,-\mu^2\right).$$

The determining factor for continuum behaviour or otherwise may be judged by comparing terms like $(I_1)_s$ and $(I_1)_{s=2}$, in the r.h.s. of the momentum equation (2·4). In fact

$$(I_1)_s/(I_1)_{s=2} \propto c_\perp^{4(s-2)/(s-1)} \propto T_0^{2(s-2)/(s-1)} \propto r^{-8(s-2)/3(s-1)} \quad \text{for large } r.$$

In order to obtain a non-uniform expansion from the continuum solution a factor at least of order r, the difference between $r^{-\frac{4}{3}}$ and $r^{-\frac{7}{3}}$ (see (3·4)), must be found. This requires

$$8(s-2)/3(s-1) > 1, \quad \text{or} \quad s > \tfrac{13}{5}.$$

We now see how special $s = 2$ is in the sequence of integral values of s. Not only does it require a cut-off in the collision integral to counteract the vast cross-section of interaction, but even with this cut-off a continuum behaviour results. For $s > \frac{13}{5}$, and thus for $s = 3$ and higher integral values, a non-uniformity in the continuum expansion occurs, and a rescaling after the manner of Hamel & Willis becomes necessary.

We have examined the case of a gas consisting of only one kind of particle, and find that the same type of integral occurs as in our own work. The critical value of s must therefore also be $\frac{13}{5}$ for jets of Hamel & Willis's type.

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