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The expansion of a plasma from a spherical source into a vacuum

Part 2. Partially-ionized flow

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This paper introduces the effects of partial ionization and recombination into the steady supersonic spherical source expansion of a plasma. It presents an analytical solution for a problem previously solved only numerically, using moment equations derived directly from the Boltzmann equation under the assumption of an ellipsoidal distribution function. The recombination is assumed collisional–radiative, as considered by Bates, Kingston & McWhirter (1962*a*, *b*) and an approximate rate constant is derived from their results. The plasma is taken to be optically thick. The results show that, in a highly ionized plasma, the species maintain equal isotropic temperatures for the main part of the flow, with the degree of ionization tending to a finite limit. Recombinations, however, remain sufficiently frequent to keep the temperature above that in an expansion with no recombination. In a weakly ionized plasma the temperature of the heavy particles is almost unaffected by recombinations and falls adiabatically, although the electron temperature remains high. The analytical results are compared with the numerical results of Chou & Talbot (1967).

1. Introduction

The steady spherically symmetric source flow expansion into a vacuum of a partially ionized gas, where recombination and ionization effects are included, was investigated numerically by Chou & Talbot and by Gol'dfarb, Kostygova & Luk'yanov (1969). This work considers the exactly similar problem, but attempts to solve it analytically, in order to give further insight into the behaviour of the flow quantities. It is an extension of Goldfinch & Pack (1971), where the plasma was assumed fully ionized throughout the flow. The analytical techniques adopted are those used by Cooper & Bienkowski (1966) when solving for the expansion of a neutral gas mixture, namely matched asymptotic expansion. The equations used are essentially the same as those of Chou & Talbot, but somewhat simpler expressions for the recombination rates are derived from values computed by Bates *et al.*, and the collisional changes are calculated from the Boltzmann collision integral, as in Goldfinch & Pack (1971), rather than from a relaxation model. To obtain a closed set of equations, and to evaluate the collision integrals, we assume an ellipsoidal distribution function as proposed by Holway

(1965), and discussed in connexion with hypersonic source-flow expansions by Hamel & Willis (1966).

The recombination process in the plasma is mainly three-body and such that when an electron and ion combine to form a neutral atom, part of the energy liberated is given to further electrons colliding with the atom, and part is radiated away. Of the radiated energy, some leaves the plasma altogether and some is absorbed by neutral atoms. These then become excited, and will either re-radiate the energy, or lose it to a colliding electron, Thus, eventually the energy produced in a recombination will partly diffuse through the plasma, and leave it and partly be absorbed by the electrons, the relative proportions depending on the type of plasma. Chou & Talbot considered the limiting cases of an 'optically thin' plasma, where all the radiated energy leaves the plasma, and of an 'optically thick' plasma, where all the energy is absorbed by the electrons. Laboratory plasmas and plasmas in jet flows are generally optically thick to some forms of radiation, and optically thin to other forms. This is discussed by Byron *et al.* (1962), Chou & Talbot find that the results are qualitatively similar for the thin and thick flows, so the optically thick case we investigate here for simplicity may be taken to be a fairly good approximation to the behaviour of all plasma expansions. Gol'dfarb *et al.* find very little difference in their results for plasmas optically thick only to resonance radiation (a typical laboratory condition), and plasmas optically thick to all radiation.

Two cases are investigated, one with high and one with low initial ionization.

2. Equations

As in the former problem of a fully ionized plasma with no recombination, (Goldfinch & Pack), we derive our equations from the steady Boltzmann equations for each species

$$\mathbf{v}\,.\,\nabla_{\mathbf{x}} g_s + \frac{e_s \mathbf{E}_s}{m_s}\,.\,\nabla_{\mathbf{v}} g_s = \left(\frac{\partial g_s}{\partial t}\right)_c,$$

using $g_s = g_s(\mathbf{x}, \mathbf{v})$ as the distribution function for the sth species ($s = E, I, A$, representing electrons, ions and atoms) with $\mathbf{v}$, e_s, m_s, $\mathbf{E}_s$ particle velocity, charge, mass and acting electric field respectively. $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{v}}$ represent gradient operators in the $\mathbf{x}$-space and $\mathbf{v}$-space, and '$(\partial g_s/\partial t)_c$' represents the change in g_s per unit time arising from collisions with other particles and is taken here as the Boltzmann collision integral. For any species property ϕ_s, the total rate of change of ϕ_s induced by encounters per unit volume is

$$\int \phi_s \left(\frac{\partial g_s}{\partial t}\right)_c d\mathbf{v} + \text{any change due to recombinations.}$$

From the Boltzmann equations we thus get the moment equations governing the rate of change of species properties, and, by summing over the three species, moment equations for the plasma as a whole. The equations we use here are those giving the rate of change of plasma mass, momentum and energy; electron number density, energy and radial stress; and ion radial stress. Plasma mass, momentum and energy are collisional invariants, but for the other equations it is necessary to assume a form for g_s in order to evaluate the collision integrals. As in the problem of fully ionized flow, an ellipsoidal weight function is assumed, giving zero heat flux and a closed set of equations,

$$g_s = n_s \left(\frac{m_s}{2\pi k T_{s\parallel}}\right)^{\frac{1}{2}} \left(\frac{m_s}{2\pi k T_{s\perp}}\right) \exp\left\{-\frac{m_s(v_\theta^2+v_\phi^2)}{2kT_{s\perp}} - \frac{m_s(v_r-u_s)^2}{2kT_{s\parallel}}\right\}.$$

This form of distribution also implies a diagonal stress tensor:

$$\mathbf{P}_s = P_{s\parallel}\hat{\mathbf{r}}\hat{\mathbf{r}} + P_{s\perp}(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}),$$

with $$P_{s\parallel} = n_s kT_{s\parallel}, \quad P_{s\perp} = n_s kT_{s\perp}, \quad T_s = \tfrac{1}{3}(T_{s\parallel} + 2T_{s\perp}).$$

Here $T_\parallel$ and $T_\perp$ represent translational temperatures associated with random motion along and transverse to the streamlines, respectively. k is the Boltzmann constant, n_s and u_s are number density and mean speed (necessarily radial) of the sth species, $\mathbf{v} = (v_r, v_\theta, v_\phi)$ is the particle velocity expressed in spherical polar co-ordinates. By using this distribution, the Boltzmann collision integrals are evaluated as in Goldfinch & Pack, assuming Coulomb interactions for charged-charged particle collisions, and an inverse fifth-power law for interactions between neutral atoms and charged particles.

Chou & Talbot assume that quasi-neutrality ($n_E = n_I$) is maintained throughout the flow region of interest, and that an electric field exists which keeps the net electrical current zero, so that $u_E = u_I$. Provided that the Debye length is small compared with a characteristic length, this is a valid assumption, and we have made it here. We have also assumed that the atom and ion temperatures and macroscopic velocities are equal, since only one collision is required to equilibrate these.

Into the plasma and electron energy and electron $\hat{r}\hat{r}$-stress equations we have to introduce terms representing the rate of change of these quantities due to recombination. If we assume that the energy released in recombination is distributed isotropically among the electrons, then the rate of change of electron $\hat{r}\hat{r}$-stress is one-third of the total rate of increase in electron energy or $\frac{1}{3}\chi(-\partial n_E/\partial t)$ where χ is the ionization potential of the atom. $-\partial n_E/\partial t$ is the rate of decrease in number density of the electrons, i.e. the rate at which recombinations occur. From the electron and plasma mass conservation equations we have

$$-\frac{\partial n_E}{\partial t} = un\frac{df}{dr},$$

where $n = n_I + n_A = n_E + n_A$, and $f = n_E/n$, the degree of ionization, and if α and S are the recombination and ionization rate coefficients computed by Bates *et al.*

$$\frac{\partial n_E}{\partial t} = -\alpha n_E^2 + S n_E n_A,$$

giving $$\frac{df}{dr} = -\alpha\frac{nf^2}{u} + S\left(\frac{nf}{u} - \frac{nf^2}{u}\right).$$

S, for the source values taken here, is negligible when compared with α, and decreases far more rapidly with distance from the source, and so is neglected everywhere.

Below we give the resulting governing equations for the problem, where variables have been non-dimensionalized with respect to the corresponding source quantities. If dimensional quantities are marked by a dash, and source values by a star we have:

$$r' = r^*r, \quad u' = u^*u, \quad T' = T^*T \quad \text{(for all temperatures)},$$

$$n' = n^*n, \quad n'_E = n^*n_E, \quad \alpha' = \frac{\alpha u^*}{r^*n^*}, \quad \chi' = kT^*\chi,$$

$$c^2 = \text{the non-dimensional parameter, } \frac{m_A u^{*2}}{kT^*},$$

$$\lambda = \text{the non-dimensional parameter, } \frac{n^*r^*e^4}{u^*(kT^*)^{\frac{3}{2}}\,16\pi^2\epsilon_0^2\sqrt{m_E}}.$$

Plasma continuity:
$$r^2un = 1. \tag{1.1}$$

Electron number density:
$$\frac{df}{dr} = -\alpha\frac{nf^2}{u}. \tag{1.2}$$

Plasma momentum:

$$c^2u\frac{du}{dr} - \frac{1}{u}\frac{du}{dr}(T_{I\parallel} + fT_{E\parallel}) + \frac{d}{dr}(T_{I\parallel} + fT_{E\parallel}) - \frac{2}{r}(T_{I\perp} + fT_{E\perp}) = 0. \tag{1.3}$$

Plasma energy:

$$c^2u^2 + 3(T_{I\parallel} + fT_{E\parallel}) + 2(T_{I\perp} + T_{E\perp}) + 2\chi f = \text{constant}. \tag{1.4}$$

Electron energy:

$$3uf\frac{dT_E}{dr} + 2f\frac{du}{dr}T_{E\parallel} + 4uf\frac{T_{E\perp}}{r} + 3u\frac{df}{dr}T_E + 2\chi u\frac{df}{dr}$$
$$= -\Lambda_1 f(1-f)\,n\,(T_E - T_I) + \Lambda_2 f^2 n\frac{T_I - T_E}{T_E^{\frac{3}{2}}}. \tag{1.5}$$

Electron $\hat{r}\hat{r}$-stress:

$$u\frac{df}{dr}T_{E\parallel} + uf\frac{dT_{E\parallel}}{dr} + 2\frac{du}{dr}fT_{E\parallel} + \left(\frac{2\chi}{3} + T_E\right)u\frac{df}{dr}$$
$$= \Lambda_3 f(1-f)n(T_{E\perp} - T_{E\parallel}) + \Lambda_4 f^2 n\frac{T_{E\perp} - T_{E\parallel}}{T_E^{\frac{3}{2}}}. \tag{1.6}$$

Ion $\hat{r}\hat{r}$-stress:

$$u\frac{df}{dr}T_{I\parallel} + uf\frac{dT_{I\parallel}}{dr} + 2\frac{du}{dr}fT_{I\parallel} = \Lambda_3 f(1-f)n(T_{I\perp} - T_{I\parallel}) + \Lambda_5 f^2 n\frac{T_{I\perp} - T_{I\parallel}}{T_I^{\frac{3}{2}}}. \tag{1.7}$$

In these equations we have used

$$\Lambda_1 = \frac{5K^{\frac{1}{2}}\pi\sqrt{m_E}\,r^*n^*}{m_A u^*}, \quad \Lambda_2 = \frac{2P_1\lambda m_E}{m_I},$$

$$\Lambda_3 = \frac{1{\cdot}688K^{\frac{1}{2}}\pi\, 2r^*n^*}{\sqrt{m_E}\, 3u^*}, \quad \Lambda_4 = \frac{2(2+\sqrt{2})\,P_1\lambda}{15},$$

$$\Lambda_5 = \frac{2\sqrt{2}\,P_1\lambda}{15}\left(\frac{m_E}{m_I}\right)^{\frac{1}{2}}, \quad P_1 = 4(2\pi)^{\frac{1}{2}}\ln\Lambda.$$

K is given in appendix B. In the following work α has been taken as: $1{\cdot}75n_E T_E^{-\frac{11}{2}}$ (see appendix A).

3. Solution

If source quantities of the same order of magnitude are used in this problem as were used in the constant-f problem (*viz.* $n^* \simeq 0{\cdot}5\times 10^{23}\,\mathrm{m}^{-3}$,

$$r^* \simeq 10^{-2}\,\mathrm{m}, \quad T^* \simeq 10^4\,{}^\circ\mathrm{K}, \quad u^* \simeq \tfrac{10}{3}(kT^*/m_A),$$

we obtain the following orders of magnitude for the coefficients Λ_i:

$$\Lambda_1 = O(1) \quad \text{for a hydrogen plasma,} \quad O(\nu^{-1}) \quad \text{for an argon plasma,}$$

$$\Lambda_2, \Lambda_3, \Lambda_5 = O(\nu), \quad \Lambda_4 = O(\nu^2),$$

where $$\nu = \lambda(m_E/m_I)^{\frac{1}{2}} \simeq 800.$$

Since ν is very large, it is possible to simplify our equations by expanding the variables in inverse powers of ν, as e.g.

$$T_{E\parallel} = T_{E\parallel 0} + \nu^{-1}T_{E\parallel 1} + \nu^{-2}T_{E\parallel 2} + \dots .$$

On comparing terms of equal order in the equations, we find, for lowest order of ν^{-1} in (1.5)–(1.7), that

$$T_{I\parallel 0} = T_{I\perp 0}, \quad T_{E\parallel 0} = T_{E\perp 0}, \quad T_{E0} = T_{I0} = T_0, \text{ say}$$

(i.e. the flow is inviscid to first order and electron and ion temperatures are equal). On using T_0, (1.2)–(1.3) become, to lowest order,

$$c^2u_0^2 + 5T_0(1+f_0) + 2\chi f_0 = \text{constant} = c^2 + 5 + (5+2\chi)f^*,$$

$$c^2\frac{du_0}{dr} + r^2\frac{d}{dr}\left\{\frac{T_0(1+f_0)}{r^2u_0}\right\} = 0,$$

$$\frac{df_0}{dr} + \frac{1{\cdot}75f_0^3}{r^4u_0^3T_0^{\frac{11}{2}}} = 0.$$

We can extract the behaviour for large r from these equations as

$$u_0 = A_0 + A_1r^{-\frac{6}{13}} + \dots,$$

$$T_0 = B_1r^{-\frac{6}{13}} + \dots,$$

$$f_0 = D_0 + D_1r^{-\frac{6}{13}} + \dots,$$

where A_0 is unknown, and the other constants satisfy the equations

$$3A_0A_1c^2 = -16B_1(1+D_0), \quad \text{so} \quad A_1 < 0,$$

$$6D_1 = 13 \times 1{\cdot}75 D_0^3 / A_0^3 B_1^{\frac{11}{2}}, \quad \text{so} \quad D_1 > 0,$$

$$2c^2 A_0 A_1 + 5B_1(1+D_0) + 2\chi D_1 = 0,$$

and

$$c^2 A_0^2 + 2\chi D_0 = (c^2+5) + (5+2\chi) f^*.$$

This solution may be fitted to the argon plasma numerical solution of Chou & Talbot if B_1 is chosen as 0·75 (see figure 1).

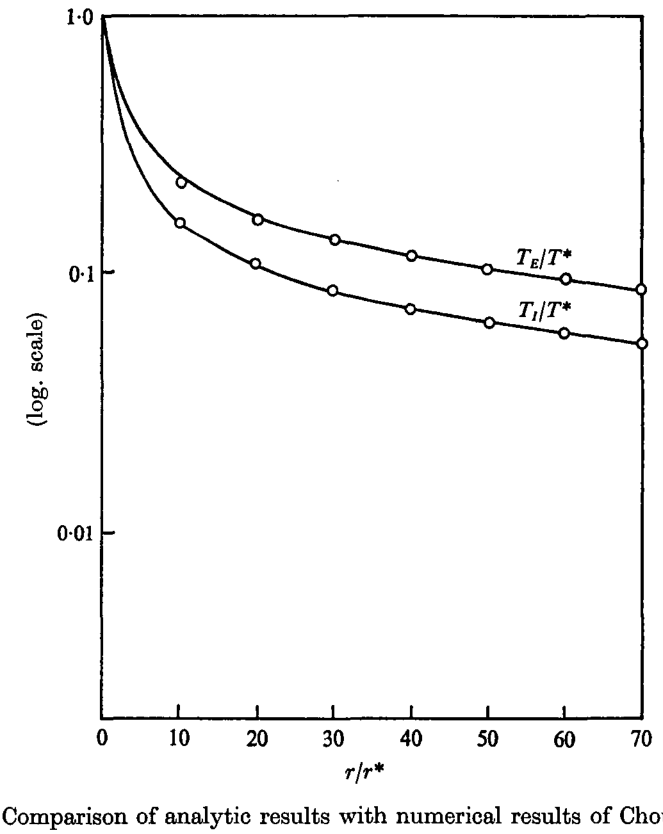

FIGURE 1. Comparison of analytic results with numerical results of Chou & Talbot: high initial ionization for argon. O, analytical values; ———, numerical results. Temperature variation.

Physically this solution shows that the plasma velocity rises to a limiting value far from the source, while the degree of ionization falls to a limiting value, and the plasma temperature continues to fall as we move outward.

Further use of the expansion in ν^{-1} in (1.1)–(1.7) shows that only the electron temperature remains isotropic to order ν^{-2}. The temperature difference and ion anisotropy are

$$(T_{E1} - T_{I1}) = \frac{\nu}{\Lambda_2}\left[\frac{(34 D_0 B_1 - 12\chi D_1)}{13 D_0^2} A_0^2 B_1^{\frac{1}{2}}\right] T_0 r^{\frac{4}{13}},$$

and

$$(T_{I\parallel 1} - T_{I\perp 1}) = \frac{\nu}{\Lambda_5}\left[\frac{6 A_0^2 B_1^{\frac{3}{2}}}{13 D_0}\right] T_0 r^{\frac{4}{13}}.$$

The expansion in ν^{-1} is only valid when

$$\frac{\nu^{-1}T_{E1}}{T_{E0}} \quad \text{and} \quad \frac{\nu^{-1}T_{I1}}{T_{I0}} < 1;$$

therefore, when $r^{\frac{4}{13}} \sim \Lambda_2$ (i.e. $r \sim \nu^{\frac{13}{4}} \sim 10^{10}$), the expansion will break down. This is a very large distance from the source, and the solution further out than this is not really of any practical interest. It is, however, possible by re-scaling the variables to continue the solution beyond this point.

Second region

For this region we use the scaled variables

$$\rho = r\nu^{-\frac{13}{4}}, \quad u = u, \quad (\tau_{E\parallel}, \tau_{E\perp}, \tau_{I\parallel}, \tau_{I\perp}) = (T_{E\parallel}, T_{E\perp}, T_{I\parallel}, T_{I\perp})\,\nu^{\frac{3}{2}},$$

$$f = f, \quad \eta = n\nu^{\frac{13}{2}}.$$

When we have expressed (1.1)–(1.7) in terms of these variables, we have the governing equations for the region (2·1)–(2.7), not reproduced here. The solutions for this region are obtained by the same methods as used in the first region, and by expanding the variables again, this time in $\nu^{-\frac{3}{2}}$, we get the first-order equations,

$$\left.\begin{aligned} f_0 &= \text{constant} = D_0, \\ u_0 &= \text{constant} = A_0, \end{aligned}\right\} \text{to match with the first region,} \tag{3.6}$$

$$\frac{df_1}{d\rho} + \frac{1{\cdot}75D_0^3}{\rho^4 A_0^3 \tau_{E0}^{\frac{11}{2}}} = 0, \tag{3.2}$$

$$c^2 A_0 \frac{du_1}{d\rho} + \frac{d}{d\rho}(\tau_{I\parallel 0} + D_0\tau_{E0}) - \frac{2}{\rho}(\tau_{I\perp 0} + D_0\tau_{E0}) = 0, \tag{3.3}$$

$$2c^2 A_0 \frac{du_1}{d\rho} + \frac{d}{d\rho}(3\tau_{I\parallel 0} + 2\tau_{I\perp 0} + 5D_0\tau_{E0}) + 2\chi\frac{df_1}{d\rho} = 0, \tag{3.4}$$

$$3\frac{d\tau_{E0}}{d\rho} + \frac{4\tau_{E0}}{\rho} + \frac{2\chi}{D_0}\frac{df_1}{d\rho} = \frac{\Lambda_2 D_0}{\nu A_0^2}\frac{(\tau_{I0} - \tau_{E0})}{\rho^2 \tau_{E0}^{\frac{3}{2}}}. \tag{3.5}$$

$$A_0 D_0 \frac{d}{d\rho}(\tau_{I\parallel 0}) = \frac{\Lambda_5 D_0^2}{\nu A_0} \times \frac{(\tau_{I\parallel 0} - \tau_{I\perp 0})}{\rho^2 \tau_{I0}^{\frac{3}{2}}}, \tag{3.7}$$

The behaviour of the variables for large ρ can be derived from these equations as

$$\left.\begin{aligned} \tau_{I\parallel 0} &\simeq E_0\rho^{-\frac{2}{3}}, \\ \tau_{I\perp 0} &\simeq \frac{E_0}{4}\rho^{-\frac{2}{3}}, \\ \tau_{E0} &\simeq F_0\rho^{-\frac{6}{13}}, \end{aligned}\right\} \quad \text{where} \quad \left\{\begin{aligned} E_0 &= \left(\frac{9}{8}\frac{D_0}{A_0^2}\frac{\Lambda_5}{\nu}\right)^{\frac{2}{3}}, \\ F_0 &= \left(\frac{13}{17}\frac{D_0^2}{A_0^3} \times 1{\cdot}75\chi\right)^{\frac{2}{13}}. \end{aligned}\right.$$

We must check that the solution of (2.1)–(2.7) for small ρ matches with the solution for large r in the first region, i.e. we must check that

$$\left.\begin{aligned} \tau_{I\parallel 0} &= \frac{B_1}{\rho^{\frac{6}{13}}}[1 + A_0' \rho^{\alpha} + \ldots] \\ \tau_{I\perp 0} &= \frac{B_1}{\rho^{\frac{6}{13}}}[1 + B_0' \rho^{\beta} + \ldots] \\ \tau_{E0} &= \frac{B_1}{\rho^{\frac{6}{13}}}[1 + C_0' \rho^{\gamma} + \ldots] \end{aligned}\right\}, \quad \text{where} \quad \alpha, \beta, \gamma > 0,$$

satisfy equations (3.2)–(3.7) for small ρ. We find that this is so, provided we take whichever is smallest of α, β, γ to be $\frac{4}{13}$.

When we look at terms of $O(1)$ in (2.6), we discover that the solution in this region also breaks down as ρ (and r) increases. This breakdown occurs when $\rho \sim \nu^{\frac{13}{8}}$, i.e. $r \sim \nu^{\frac{39}{8}} \sim 10^{15}$, a very great distance from the source.

It is interesting to note that, if we repeat the rescaling technique to give equations for a third region, we find the electron temperature becoming anisotropic, and the parallel temperature freezing while the perpendicular falls as r^{-1}. However it appears not to be possible to obtain solutions for $T_{I\parallel}$ and $T_{I\perp}$ in this region, as coefficients in the plasma momentum and energy equations become unsuitable for an expansion in inverse powers of ν.

Concluding remarks

When the initial ionization is high, Chou & Talbot's computed results appear to be borne out well by these analytical results. Collisions appear to be sufficient to maintain isotropy and equal ion-electron temperatures for a large distance from the source, although the energy released in recombinations produces a slower rate of fall in temperature than is the case with no recombination. Far from the source, however, the ion temperature is the first to become anisotropic (see Cooper & Bienkowski, where the heavier neutral species is the first to become anisotropic), and collisions with the electrons decrease, so that the ion temperature begins to fall faster than that of the electrons. Still farther from the source, the electron temperature becomes anisotropic, and the parallel temperature freezes; the behaviour of the ion temperature at this point is unfortunately unclear.

As expected, the velocity and ionization both approach frozen values only a short distance from the source.

4. Low initial ionization

For flow where the source ionization is small, the governing equations are better written in terms of the variable f', where $f = f^* f'$, so that $f' = 1$ at the source. The procedure for determining the flow is then exactly similar to that used for large initial ionization. The solution, however, shows some differences: the ion temperature falls as $r^{-\frac{4}{3}}$ throughout the flow region calculated (i.e. falls as the temperature falls at first in a neutral gas, or in the problem of a plasma where recombination is not accounted for). The electron temperature, however, although falling for a very short distance close to the source as $r^{-\frac{4}{3}}$, then begins to fall less rapidly, so that it is very much greater than the ion-temperature. Eventually, the electron parallel temperature freezes, and the perpendicular temperature falls as r^{-1}.

Solution

The governing equations now are as follows, where f' has been written as f for simplicity:

$$r^2un = 1, \tag{4.1}$$

$$\frac{df}{dr} + \frac{1{\cdot}75f^{*2}f^3}{r^4u^3T_E^{\frac{11}{2}}} = 0, \tag{4.2}$$

$$c^2u\frac{du}{dr} + \frac{d}{dr}(T_{I\parallel} + f^*fT_{E\parallel}) - \frac{1}{u}\frac{du}{dr}(T_{I\parallel} + f^*fT_{E\parallel}) - \frac{2}{r}(T_{I\perp} + f^*fT_{E\perp}) = 0, \tag{4.3}$$

$$c^2u^2 + 3(T_{I\parallel} + f^*fT_{E\parallel}) + 2(T_{I\perp} + f^*fT_{E\perp}) + 2\chi f^*f = \text{constant}, \tag{4.4}$$

$$3uf\frac{dT_E}{dr} + 2\frac{du}{dr}fT_{E\parallel} + 4uf\frac{T_{E\perp}}{r} + 3u\frac{df}{dr}T_E + 2\chi u\frac{df}{dr}$$
$$= -\Lambda_1 f(1-f^*f)\,n(T_E - T_I) - \Lambda_2 f^*f^2n\frac{T_E - T_I}{T_E^{\frac{3}{2}}}, \tag{4.5}$$

$$uT_{E\parallel}\frac{df}{dr} + uf\frac{dT_{E\parallel}}{dr} + 2fT_{E\parallel}\frac{du}{dr} + (2\chi/3 + T_E)\frac{df}{dr}$$
$$= \Lambda_3 f(1-f^*f)\,n(T_{E\perp} - T_{E\parallel}) + \Lambda_4 f^*f^2n(T_{E\perp} - T_{E\parallel})/T_E^{\frac{3}{2}}, \tag{4.6}$$

$$uT_{I\parallel}\frac{df}{dr} + uf\frac{dT_{I\parallel}}{dr} + 2fT_{I\parallel}\frac{du}{dr} = \Lambda_3 f(1-f^*f)\,n(T_{I\perp} - T_{I\parallel}) + \Lambda_5 f^*f^2n\frac{T_{I\perp} - T_{I\parallel}}{T_I^{\frac{3}{2}}}. \tag{4.7}$$

If we take as a typical case $f^* \sim \nu^{-\frac{1}{2}}$, then the terms appearing on the r.h.s. of (4.5)–(4.7) have the order of magnitude,

(4.5) $O(1)$ and $O(\nu^{\frac{1}{2}})$ for hydrogen, $O(\nu^{-1})$ and $O(\nu^{\frac{1}{3}})$ for argon,

(4·6) $O(\nu)$ and $O(\nu^{\frac{3}{2}})$,

(4.7) $O(\nu)$ and $O(\nu^{\frac{1}{2}})$.

We see that in this case the neutral collisions predominate in changing the ion $\hat{r}\hat{r}$-stress, but that the Coulomb collisions remain more important in changing the electron energy.

Expanding the quantities in powers of $\nu^{-\frac{1}{2}}$ we find that

$$T_0 = T_{E\parallel0} = T_{E\perp0} = T_{I\parallel0} = T_{I\perp0},$$

and two equations governing u_0 and T_0 that occur in the problem of a neutral gas, *viz.*

$$\left.\begin{aligned} c^2u_0\frac{du_0}{dr} + \frac{dT_0}{dr} - \frac{T_0}{u_0}\frac{du_0}{dr} - \frac{2T_0}{r} &= 0,\\ c^2u_0^2 + 5T_0 &= c^2 + 5, \end{aligned}\right\}$$

with solution $u_0^2T_0^3 = 1/r^4$.

For large r the solutions become

$$\left.\begin{aligned} u_0 &= K_0^{-3} - \frac{5K_0^{-1}}{2c^2} r^{-\frac{4}{3}} + O(r^{-\frac{8}{3}}) \\ T_0 &= K_0^2 r^{-\frac{4}{3}} + O(r^{-\frac{8}{3}}), \end{aligned}\right\} \quad \text{where} \quad K_0 = \left(\frac{c^2}{c^2+5}\right)^{\frac{1}{6}}.$$

We now consider (4·2). Knowing u_0 and T_0 for large r, we may rewrite this, for large r, as

$$\frac{1}{f^3}\frac{df}{dr} = -\frac{1{\cdot}75 f^{*2}}{K_0^2} r^{\frac{10}{3}} + \text{lower terms in } r.$$

We see that, although the r.h.s. of this is $O(\nu^{-1})$ near the source, it increases in size rapidly with r, and, when $r \sim \nu^{\frac{3}{10}} \simeq 7{\cdot}5$, it becomes $O(1)$. An expansion of f in $\nu^{-\frac{1}{2}}$ here appears to be inapplicable, so we take

$$\left.\begin{aligned} u &= K_0^{-3} + u_{10}\nu^{-\frac{1}{2}} + u_{20}\nu^{-1} + \dots, \\ T_E &= K_0^2 r^{-\frac{4}{3}} + T_{E10}\nu^{-\frac{1}{2}} + T_{E20}\nu^{-1} + \dots, \end{aligned}\right\}$$

and use these in the original equation (4.2):

$$\frac{1}{f^3}\frac{df}{dr} = -\frac{1{\cdot}75 f^{*2}}{K_0^2} r^{\frac{10}{3}} + O(r^2) + O(\nu^{-\frac{3}{2}}).$$

This gives us

$$f^{-2} \simeq \text{constant} + \frac{0{\cdot}8 f^{*2}}{K_0^2} r^{\frac{13}{3}}.$$

If $r \gg \nu^{\frac{3}{13}} \simeq 4{\cdot}5$, we may ignore the constant term, and so we have, for large r,

$$f \simeq (K_0^2/0{\cdot}8 f^{*2})^{\frac{1}{2}} r^{-\frac{13}{6}} \sim \nu^{\frac{1}{2}} r^{-\frac{13}{6}}.$$

When we consider further terms in the expansion in $\nu^{-\frac{1}{2}}$, we find that, although the dominance of the neutral-ion collision term in (4.7) produces a breakdown in the expansion, an earlier breakdown is occasioned by the term representing the gain in energy from recombinations in equation (4.5). We find that

$$\nu^{-\frac{1}{2}}\frac{(T_{I1} - T_{E1})}{T_0} = \frac{u_0\sqrt{T_0}}{f^2}\frac{df}{dr}(2\chi u_0 + 3u_0 T_0) \times \frac{1}{f^*\Lambda_2} \sim \nu^{-\frac{1}{2}} r^{\frac{15}{6}},$$

for large r, and the expansion breaks down when $r \simeq 6$ in a hydrogen plasma. (For an argon plasma, it is more accurate to take the first-order equation from (4·5), as

$$2\chi u_0 \frac{df_0}{dr} = \frac{\Lambda_2 f^* f_0^2}{u_0 r^2} \times \frac{(T_{I0} - T_{E0})}{T_{E0}^{\frac{3}{2}}}.$$

The first-order solutions are then, for large r,

$$T_{I0} = K_0^2 r^{-\frac{4}{3}}, \quad T_{E0} = K_1 r^{-\frac{2}{5}}, \quad u_0 = K_0^{-3},$$

$$f_0 = A_0 + \left(\frac{2{\cdot}2 f^{*2} A_0^3}{K_0 K_1^{\frac{11}{2}}}\right) r^{-\frac{4}{5}}.$$

This is precisely the solution for the second region in the hydrogen problem and further solution is the same for both plasmas.)

From this point the solution is continued as in § 3 by scaling the variables appropriately, and solving the new equations. We find the first-order solution for this next region to be

$$u_0, f_0 \sim \text{constants}\left(u_0 = K_0^{-3}, f_0 = \frac{K_1^5 \Lambda_2 (c^2+5)^{\frac{1}{2}}}{3{\cdot}5 f^* c\chi} = A_0, \text{ say.}\right),$$

$$T_{I0} = K_0^2 r^{-\frac{4}{3}}, \quad T_{E0} = K_1 r^{-\frac{2}{5}},$$

$$T_{E\perp 0} = T_{E\parallel 0}, \quad T_{I\perp 0} = T_{I\parallel 0},$$

for large ρ, where $\rho = r\nu^{-\frac{2}{5}}$. K_1 is an undetermined constant, and the other constants are determined by matching with the first region of flow.

When $r \sim \nu$ this solution also breaks down, and further rescaling is necessary to produce equations governing a third region still further from the source. The first-order asymptotic behaviour of the variables in this new region is

$$u_0, f_0 \sim \text{constants}\,(u_0 = K_0^{-3}, \quad f_0 = A_0)$$

$$T_{I\parallel 0} = T_{I\perp 0} = K_0^2 r^{-\frac{4}{3}}, \quad T_{E\parallel 0} = T_{E\perp 0} = \nu^{-\frac{4}{13}} D_0 r^{-\frac{6}{13}}$$

where
$$D_0 = \left(\frac{1{\cdot}3\chi\nu^{-\frac{2}{75}}(\nu f^{*2}) A_0^2}{K_0^{-9}}\right)^{\frac{2}{13}},$$

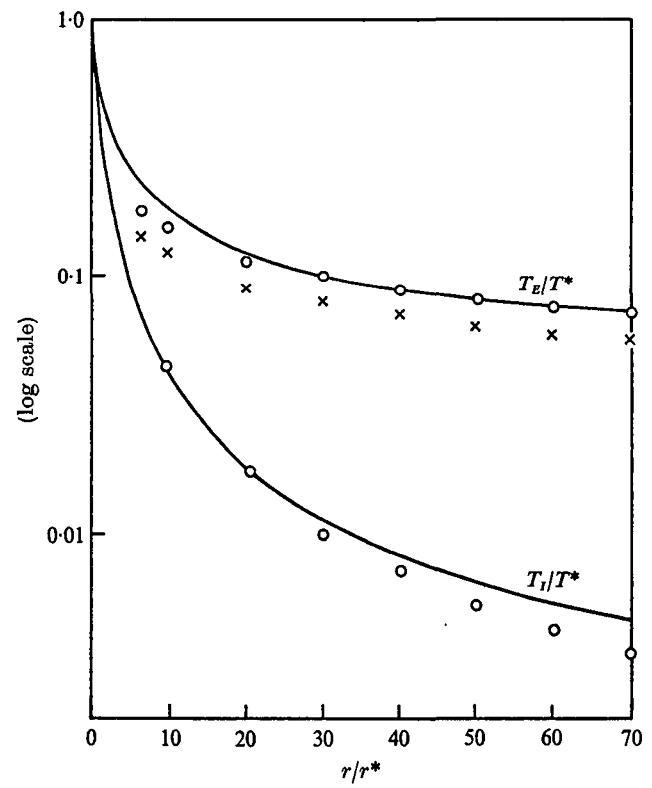

FIGURE 2. Comparison of analytic results with numerical results of Chou & Talbot: low initial ionization. Analytical results: ○, argon; ×, hydrogen. ———, numerical results.

where the constants have again been determined by matching with the previous region. A breakdown of this solution occurs when $r \sim \nu^3$, and it is in the region out from this that the parallel electron temperature $T_{E\parallel}$ freezes. Equipartition of ion temperature continues, but we no longer find $T_{E\parallel 0} = T_{E\perp 0}$; rather,

$$T_{E\parallel 0} = \nu^{\frac{17}{13}}(E_0 + E_1 r^{-1}), \quad T_{E\perp 0} = \nu^{\frac{17}{13}}(\tfrac{1}{2}E_1 r^{-1})$$

where
$$E_1 = \frac{2\Lambda_4 f^* \nu^{\frac{1}{2}} A_0}{K_0^{-6} E_0^{\frac{9}{2}}} \quad \text{for large } p = \nu^{-3} r.$$

This solution appears not to break down as p increases, but we have not investigated this thoroughly; the behaviour of T_{I0} in this region must be extracted from the equations before it can be completely checked.

Figure 2 compares the temperatures of argon and hydrogen, where K_1 has been taken as 0·26 for argon to match, with Chou & Talbot's results, and 0·33 for hydrogen.

5. Discussion

The most obvious difference between the low and high ionization expansions is the very much more rapid fall of the heavy particle temperature in the former. The greater ratio of heavy to light particles means that, although each electron will lose in collisions about the same average energy in both cases, the total energy gain, and in particular the average gain per particle, will be greatly reduced for the heavy particles. They are in fact almost independent of electron collisions. The lower rate of recombination in the less highly ionized plasma will slightly increase the rate at which the electron temperature falls. A lower ion temperature increases the frequency of ion-ion collisions, so that the ion anisotropy is delayed beyond the onset of the electron anisotropy.

Figures 1 and 2 compare the analytical results for argon and hydrogen with Chou & Talbot's numerical results for argon. The electron temperature is greater in argon than in hydrogen. This results from the inverse dependence on the ion mass of the energy exchange rate in electron-ion collisions.

After this paper had been written, a paper was presented at the Seventh International Symposium on Rarefied Gas Dynamics by R. B. Frazer, L. Talbot and F. Robben of the University of California, describing experimental measurements of flow quantities in a free jet expansion of very weakly ionized argon plasma. Their results showed a source-like flow supporting the use of a source-flow model for the centre-line properties of a plasma freejet. The centre-line density and velocity behaved isentropically, but the flow became anisotropic with the temperature raised above its isentropic value.

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Appendix A. Recombination rate coefficient

In a plasma where all radiation is re-absorbed, Bates *et al.* give the following values for the dimensional recombination rate coefficient α':

$$\alpha' = [a_r - \beta(1)] + n_E k_t,$$

with the values of k_t and $[a_r - \beta(1)]$ shown in table 1.

T_E(°K)	250	500	1000	2000	4000	8000	16000
k_t(m^6 sec^{-1})	2.6^{-31}†	$8{\cdot}8^{-33}$	$2{\cdot}9^{-34}$	$8{\cdot}4^{-36}$	$1{\cdot}9^{-37}$	$2{\cdot}4^{-39}$	$9{\cdot}1^{-41}$
$[a_r - \beta(1)]$ (m^3 sec^{-1})	$2{\cdot}4^{-18}$	$1{\cdot}4^{-18}$	$7{\cdot}7^{-19}$	$4{\cdot}0^{-19}$	$1{\cdot}5^{-19}$	$1{\cdot}7^{-20}$	$2{\cdot}4^{-21}$

† Superfixes indicate the power of 10 by which the number is to be multiplied.

TABLE 1

Provided $T_E^{\frac{1}{4}} < 10^2 \times n_E$, the term $[a_r - \beta(1)]$ in α' is negligible in comparison with $n_E k_t$. This condition is satisfied when the recombination is important in both problems.

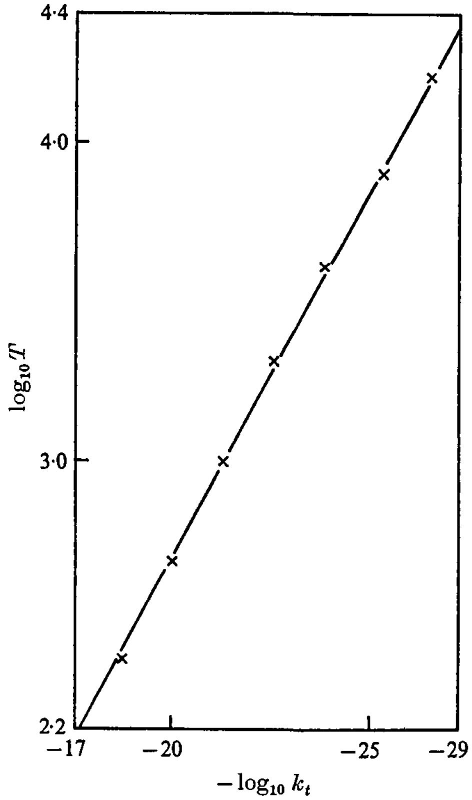

FIGURE 3. Comparison of analytical formula with numerical values of Bates, *et al.* ———, analytical formula; ×, numerical values.

Figure 3 shows a logarithmic scale graph of k_t against T_E. The points are very nearly collinear, and a line of slope $-\frac{11}{2}$, with k_t-axis intersection at $1{\cdot}12\times10^{-17}$, is obtained as a best fit. α' may then be taken, to within a very reasonable error, as

$$1{\cdot}12\times10^{-17} n_E T_E^{-\frac{11}{2}}\,\mathrm{m^3\,sec^{-1}}.$$

The above values of α' from Bates *et al.* are only valid if

$$n_E \not> 10^{8+\omega}\,\mathrm{m^{-3}},$$

where $T_E = 10^3\times 2^{\omega}\,{}^\circ\mathrm{K}$. This condition ensures a quasi-equilibrium distribution of population through the excited levels, and appears to be satisfied by the solutions for both low and high initial ionization.

S', the ionization rate coefficient, may be approximated in a similar way by

$$S' = 10^{-57} T_E^{\frac{11}{2}}\,\mathrm{m^3\,sec^{-1}}.$$

It is easily seen that $S'n_A$ is very much smaller than $\alpha' n_E$ with the source quantities chosen.

Appendix B. Collision integrals for neutral-charged collisions

Here the notation follows that of Goldfinch & Pack.

The interaction between a charged particle and a neutral particle is assumed to obey a fifth-power or Maxwell molecule law. In this case, it is unnecessary to assume a form for the distribution function in order to evaluate the collision integrals. Particle dynamics show the expression $gbdbd\epsilon$ to be independent of g, i.e. $gbdbd\epsilon = Y(\chi)\,d\chi\,d\epsilon$, so that the integrals with regard to $\mathbf{v}_A$ and $\mathbf{v}_B$ are simple averages of the velocity or the square of the velocity.

The $\hat{r}\hat{r}$-stress integral

This is the $\hat{r}\hat{r}$-component of

$$\begin{aligned}
&\int f_A f_B(\tilde{\mathbf{V}}_B\tilde{\mathbf{V}}_B-\mathbf{V}_B\mathbf{V}_B)\,gb\,db\,d\epsilon\,d\mathbf{v}_A\,d\mathbf{v}_B,\\
&\quad= 4\pi\frac{M_{AB}}{m_B}\left\{\int_0^\pi Y(\chi)\{1-\cos\chi\}\,d\chi\int f_A f_B V_{Br} g_r\,d\mathbf{v}_A\,d\mathbf{v}_B\right.\\
&\qquad-\frac{M_{AB}}{2m_B}\int_0^\pi Y(\chi)\{1-\cos\chi\}^2\,d\chi\int f_A f_B g_r^2\,d\mathbf{v}_A\,d\mathbf{v}_B\\
&\qquad\left.+\frac{M_{AB}}{4m_B}\int_0^\pi Y(\chi)\{1-\cos^2\chi\}\,d\chi\int f_A f_B(g_\theta^2+g_\phi^2)\,d\mathbf{v}_A\,d\mathbf{v}_B\right\}\\
&\quad=\left(\frac{K}{M_{AB}}\right)^{\frac{1}{2}}\frac{4\pi M_{AB}}{m_B}\left[0{\cdot}422\frac{M_{AB}}{m_B}n_A n_B(u_A-u_B)^2+\frac{M_{AB}n_B}{m_A m_B}(0{\cdot}204P_{\parallel A}\right.\\
&\qquad\left.+0{\cdot}218P_{\perp A})+0{\cdot}218\frac{M_{AB}n_A}{m_B^2}(P_{\perp B}-P_{\parallel B})-0{\cdot}422\frac{M_{AB}n_A}{m_A m_B}P_{\parallel B}\right],
\end{aligned}$$

where $$M_{AB}=\frac{m_A m_B}{m_A+m_B},\quad V = v-u.$$

The values of the integrals,

$$\int_0^\pi Y(\chi)(1-\cos\chi)\,d\chi, \quad \text{and} \quad \int_0^\pi Y(\chi)(1-\cos^2\chi)\,d\chi,$$

are given in Chapman & Cowling (1960) as

$$0{\cdot}422\left(\frac{K}{M_{AB}}\right)^{\frac{1}{2}}, \quad \text{and} \quad 0{\cdot}436\left(\frac{K}{M_{AB}}\right)^{\frac{1}{2}}$$

where K is the force constant: this is taken to be: $4\times10^{-57}\,\mathrm{J\,m^4}$ for electron–atom collisions in hydrogen, and $4{\cdot}9\times10^{-60}\,\mathrm{J\,m^4}$ in argon (see Chou & Talbot).

The energy integral

This is the trace of

$$\int f_A f_B(\tilde{\mathbf{V}}_B\tilde{\mathbf{V}}_B-\mathbf{V}_B\mathbf{V}_B)\,gb\,db\,d\epsilon\,d\mathbf{v}_A\,d\mathbf{v}_B,$$

$$=\left(\frac{K}{M_{AB}}\right)^{\frac{1}{2}}\frac{4\pi M_{AB}}{m_B}\left[0{\cdot}422\frac{M_{AB}}{m_B}n_An_B(u_A-u_B)^2+1{\cdot}266\frac{M_{AB}}{m_Am_B}(n_BP_A-n_AP_B)\right].$$

When the appropriate species A and B are chosen, the above results reduce to the expressions shown in (1.5)–(1.7), correct to first order in m_E/m_I.

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